# Linear Algebra & Geometry

## LECTURE 8

- Linear independence of sets of vectors
- Bases of a vector space

**Definition**

Let $V$ be a vector space over a field $\mathbb{K}$ and let $S = \{v_1, \dots, v_n\}$ be a (finite, obviously) set of vectors from $V$. $S$ is said to be *linearly independent* iff

$$(\forall a_1, a_2, \dots, a_n \in \mathbb{K})(a_1 v_1 + a_2 v_2 + \dots + a_n v_n = \Theta \Rightarrow a_1 = a_2 = \dots = a_n = 0)$$

If $S$ is not linearly independent, $S$ is said to be *linearly dependent*.

**Remark.**

This definition may be confusing: it is NOT that the linear combination with all zero coefficients is $\Theta$ (which is trivially true for any set of vectors, linearly independent or not) but the other way around, the condition for a set to be linearly independent is that the **only** way to make a linear combination of its vectors $\Theta$ is to make all coefficients 0.

**Examples.**

1. Is $S = \{(1,0,1), (1,1,0), (0,1,1)\}$ linearly independent in $\mathbb{R}^3$? Suppose $a(1,0,1) + b(1,1,0) + c(0,1,1) = (0,0,0)$. This means $a + b = 0, b + c = 0, a + c = 0$. Subtracting the second equation from the first we get $a - c = 0, \text{i.e.}, a = c$. replacing $a$ with $c$ in the third we get $2c = 0$ hence, $c = 0$. This easily implies that $b = a = 0$. The answer is YES.
2. The empty set $\emptyset$ is linearly independent.

**Examples.**

3. Is $S = \{(1,0,-1),(1,1,0),(0,1,1)\}$ linearly independent in $\mathbb{R}^3$? The same strategy:
   Suppose $a(1,0,-1) + b(1,1,0) + c(0,1,1) = (0,0,0)$. This means

   $$\begin{cases} a+b=0 \\ b+c=0 \\ -a+c=0 \end{cases} \quad (e_1+e_3) \rightarrow \begin{cases} b+c=0 \\ b+c=0 \\ -a+c=0 \end{cases} \quad (e_1-e_2) \rightarrow \begin{cases} 0=0 \\ b+c=0 \\ -a+c=0 \end{cases}$$

   $\rightarrow a = c, b = -c$ and no restrictions on $c$. Putting $a = c = 1$ and $b = -1$ we obtain non-zero coefficients for our linear combination.
   Conclusion: NO, the set is linearly **dependent**.

**Notice:** In example 3, $v_1 = v_2 - v_3$.

This is no coincidence.

**Theorem.**

Let $V$ be a vector space over a field $\mathbb{K}$. A set $S = \{v_1, v_2, \dots, v_n\} \subseteq \mathrm{V}$ is linearly independent iff no vector from $S$ is a linear combination of the remaining $n - 1$ vectors.

**Proof.** ($\Rightarrow$, *by contraposition*)
Suppose one of the vectors is a linear combination of the others. *Without loss of generality* we may assume that $v_n$ is the one, i.e., $v_n = a_1 v_1 + \cdots + a_{n-1} v_{n-1}$. We may write $\Theta = a_1 v_1 + \cdots + a_{n-1} v_{n-1} + (-1) v_n$. In every field $-1 \neq 0$ hence, the set $\{v_1, v_2, \dots, v_n\}$ is linearly dependent.

**Notice:**
The phrase "*Without loss of generality*" is used when instead of considering an arbitrary case (here some $v_k$) we consider a specific one (here $v_n$) because, like here,
(a) it makes no difference
(b) it simplifies notation.

**Proof.**(⇐, also by contraposition)
Suppose $\{v_1, v_2, \dots , v_n\}$ is linearly dependent, i.e. there exist coefficients $a_1, a_2, \dots , a_n$, not all of them zeroes, such that $\Theta = a_1 v_1 + \cdots + a_{n-1} v_{n-1} + a_n v_n$. Without losing generality, we may assume that $a_n \neq 0$ (we can always re-order $S$ so that the vector with the non-zero coefficient is the last one). So

$$a_n v_n = (-a_1) v_1 + \cdots + (-a_{n-1}) v_{n-1}$$

Since $a_n$, being a nonzero scalar is invertible (w.r.t. multiplication in the field $\mathbb{K}$, we have

$$v_n = (-a_1 a_n^{-1}) v_1 + (-a_2 a_n^{-1}) v_2 + \dots + (-a_{n-1} a_n^{-1}) v_{n-1}.$$ QED

**Remark.**

$S = \{v_1, v_2, \dots, v_n\}$ is linearly independent iff

$$(\forall i = 1,2,\dots,n)(v_i \notin span(S \setminus \{v_i\}).$$

**Examples.**

1. In every $\mathbb{K}^n$ the set of *unit vectors*, $\{(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,0,\dots,0,1)\}$ is linearly independent.
   $a_1(1,0,\dots,0) + a_2(0,1,0,\dots,0) + \dots + a_n(0,0,\dots,0,1) = (a_1, a_2, \dots, a_n) = (0,0,\dots,0)$ only if $a_1 = a_2 = \cdots a_n = 0$.

2. Is the set of functions $\{\sin x, \cos x, x, 2^x\}$ linearly independent in $\mathbb{R}^{\mathbb{R}}$ over $\mathbb{R}$?
First, what is the zero-vector $\Theta$ in $\mathbb{R}^{\mathbb{R}}$? Clearly it is not 0, vectors in $\mathbb{R}^{\mathbb{R}}$ are functions, not numbers. $\Theta = 0(\mathrm{x})$, the function constantly equal 0.
Second, what does $a_1 \sin x + a_2 \cos x + a_3 x + a_4 2^x = \Theta$ mean? This is equality of functions; two functions are equal if they assign the same values to the same arguments so, it means $(\forall x \in \mathbb{R})(a_1 \sin x + a_2 \cos x + a_3 x + a_4 2^x = \Theta(x) = 0)$
In particular, for $x = 0$ we get $a_1 \sin 0 + a_2 \cos 0 + a_3 0 + a_4 2^0 = 0$, i.e., $a_2 + a_4 = 0$. This means, our condition is reduced to $(\forall x \in \mathbb{R})(a_1 \sin x + \ a_3 x + a_4 2^x = 0)$. For $x = \pi$ we get $a_3 \pi + a_4 2^{\pi} = 0$ and for $x = -\pi$, $a_3(-\pi) + a_4 2^{-\pi} = 0$.
Adding the last two equations yields $a_4(2^{\pi} + 2^{-\pi}) = 0$, $a_4 = 0$ and $a_3 = 0$.$(\forall x \in \mathbb{R}) a_1 \sin x = 0$ means $a_1 = 0$. So, YES.

2. (Cont'd)
$(\forall x \in \mathbb{R})\ a_1 \sin x - a_4 \cos x + \ a_3 x + a_4 2^x = 0.$
For $x = \pi$ we get
$a_4 + a_3\pi + a_4 2^{\pi} = 0$
and for $x = -\pi$,
$a_4 + a_3(-\pi) + a_4 2^{-\pi} = 0.$
Adding the last two equations yields
$a_4(2 + 2^{\pi} + 2^{-\pi}) = 0$ hence, $\mathrm{a}_4 = 0$ and, since $a_2 = -a_4$, also $a_2 = 0$.
This reduces our condition to
$(\forall x \in \mathbb{R})\ a_1 \sin x + a_3 x = 0$
Putting $x = \pi$ we get $a_3\pi = 0$, so $a_3 = 0$ and, finally, $(\forall x \in \mathbb{R})\ a_1 \sin x = 0$ clearly implies $a_1 = 0$. The answer is YES.

3. Is the set $\{1, x, x^2, \dots, x^n\}$ a linearly independent subset of $\mathbb{R}[x]$?
(induction on $n$)
$\{1\}$ and $\{1, x\}$ are linearly independent. Suppose $\{1, x, x^2, \dots, x^n\}$ is linearly independent and $\{1, x, x^2, \dots, x^{n+1}\}$ is not. This implies that $x^{n+1} = \sum_{s=0}^{n} a_s x^s$
(**Comprehension:** Why can we claim that it is $x^{n+1}$ which is a linear combination of other vectors, not one of $1, x, x^2, \dots, x^n$?).
Differentiating both sides $n + 1$ times yields $(n + 1)! = 0$.
So, YES.

**Fact.**

Let $V$ be a vector space over a field $\mathbb{K}$. If a set $S, S \subseteq V$, is linearly independent then every subset of $S$ is linearly independent.

**Proof.**

This is obvious. If $R \subseteq S$ and $R$ is linearly dependent then a vector from $R$, say $r$, is a linear combination of other vectors from $R$. But then a vector from $S$ (namely this very vector $r$) is a linear combination of other vectors from $S$ – the same vectors, which means $S$ is linearly dependent.

**Definition.**

If a set $S, S \subseteq$ V, is linearly independent and $span(S) = V$ then $S$ is called *a basis* of V.

**Remark.**

We defined linear independence only for finite sets of vectors. It can be extended to infinite sets but in this course, we will only use finite linearly independent sets and, consequently, finite bases. Vector spaces who have finite bases are said to be *"finite-dimensional*".

**Examples.**

1. In every $\mathbb{K}^n$ the set of *unit vectors*, $S = \{(1,0,\dots,0),$ $(0,1,0,\dots,0),\dots,(0,0,\dots,0,1)\}$ is a basis.
   We know it is linearly independent. Obviously, for every vector $v = (x_1, x_2, \dots, x_n)$ we can write $v = x_1(1,0,\dots,0) + x_2(0,1,0,\dots,0), + \ \dots + x_n(0,0,\dots,0,1)$ so
   $$span(S) = V.$$
2. $\{1, x, x^2, \dots, x^n\}$ is a basis for $\mathbb{R}_n[x]$.
3. $\mathbb{R}[x]$ has no finite basis.
4. $\{1, i\}$ is a basis for $\mathbb{C}$ over $\mathbb{R}$.
5. $\{1\}, \{i\}, \{1+i\}$ are bases for $\mathbb{C}$ over $\mathbb{C}$.

**Theorem**

A set $S = \{v_1, v_2, \ldots, v_n\}$ is a basis of a vector space $V$ over $\mathbb{K}$ iff for every $v \in$ V there exist unique coefficients $a_1, a_2, \ldots, a_n$ such that $v = a_1 v_1 + a_2 v_2 + \ \ldots + a_n v_n = \sum_{s=1}^{n} a_s v_s$

**Proof.**(⇒)
The existence of $a_1, a_2, \ldots, a_n$ follows from $span(S) = V$.
Suppose there are other coefficients, $b_1, b_2, \ldots, b_n$ such that $v = \sum_{s=1}^{n} b_s v_s$. Then, $\Theta = v - v = \sum_{s=1}^{n} a_s v_s - \sum_{s=1}^{n} b_s v_s = \sum_{s=1}^{n}(a_s - b_s) v_s$. Since $S$ is linearly independent, this means $a_s - b_s = 0, s = 1,2, \ldots, n$.

(⇐) $span(S) = V$ is obvious. Suppose $a_1 v_1 + a_2 v_2 + \ \ldots + a_n v_n = \Theta$; since $0 v_1 + 0 v_2 + \ \ldots + 0 v_n = \Theta$, uniqueness of coefficients gives us $a_i = 0, i = 1,2, \ldots, n$. QED